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THE DIFFUSION OF LOAD FROM A BAR EMBEDDED IN A SEMI-INFINITE ELASTIC MEDIUM

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Abstract—The diffusion of load from a finite bar embedded in a semi-infinite elastic medium is examined. The integral equation approach embraces both projecting and totally embedded bars. A range of cases is analysed and comparative solutions have been obtained from a finite element analysis.

NOTATION

A_b	area of bar
E	Young's modulus
E_f	$E_b - E_m$
K_{ij}	influence coefficients
L_1, L_2	z-coordinates of ends of bar
L_3	z-coordinate of the external force
P	external force
R	radius of bar
R_1, R_2	$+\sqrt{(R^2+(z-c)^2)}$, $+\sqrt{(R^2+(z+c)^2)}$
c	general point defined in Fig. 1
n	$(z-c)/\lvert z-c\rvert$
p	longitudinal force in bar
q	force/unit length applied to the medium as a consequence of the diffusion of load in the bar
s	radial stress on the bar
x, y, z / r, θ, z	coordinate axes
ν	Poisson's ratio

Suffices b, m refer to the bar and the medium, respectively.

1. INTRODUCTION

LOAD diffusion from a bar to an attached flat plate has been investigated by a number of authors and their results have been comprehensively discussed elsewhere [1, 2]. For a bar embedded in a semi-infinite elastic medium an analagous three-dimensional approach, as proposed by Muki and Sternberg [3, 4], may be adopted.

Figure 1 represents, schematically, the problem as considered herein, the particular cases examined are:

(i) an embedded bar, with

$$L_1 \rightarrow \infty$$

$$L_1/(L_2-L_1) \gg 1$$

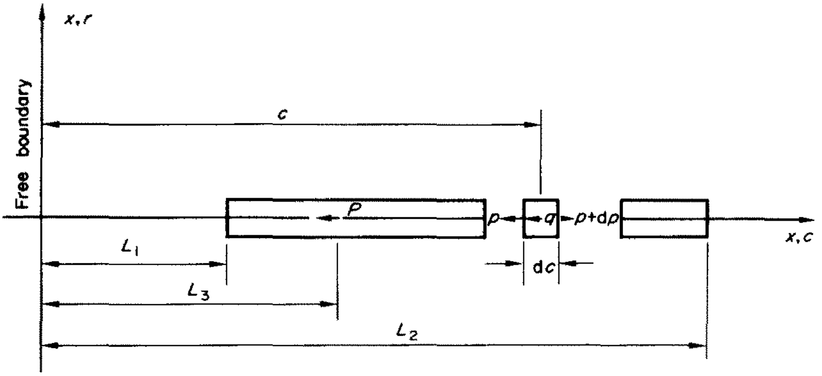

FIG. 1. Schematic representation.

and

$$L_3 = (L_1 + L_2)/2;$$

(ii) a projecting bar, with

$$L_1 = L_3 = 0.$$

2. GENERAL EQUATIONS

The bar–medium combination is initially considered to consist of a continuous medium of Young's Modulus E_m and a fictitious bar of Young's Modulus $(E_b - E_m)$. This approach follows directly from essentially two-dimensional methods applied to stiffener-sheet combinations [1, 2] and was proposed by Muki and Sternberg [3, 4].

Consideration of strain compatibility between the fictitious bar and the medium yields an integral equation. To set up this integral equation it is convenient to define influence functions $K_{ij}(x, y, z, r, c)$ where $K_{11}(x, y, z, r, c)$ and $K_{21}(x, y, z, r, c)$ represent, respectively, the longitudinal and radial strains in the medium at (x, y, z) due to a stress of magnitude $1/\pi r^2$, in the negative z-direction, acting over a circular area defined by:

$$x^2 + y^2 \leq r; \qquad z = c.$$

$K_{12}(x, y, z, r, c)$ and $K_{22}(x, y, z, r, c)$ represent, respectively, the longitudinal and radial strains in the medium at (x, y, z) due to a positive unit stress acting radially over the cylindrical area defined by:

$$x^2 + y^2 = r^2; c - \tfrac{1}{2}\,\mathrm{d}c < z < c + \tfrac{1}{2}\,\mathrm{d}c.$$

The idealized forces applied to the medium by an element of the fictitious bar, of length dc, comprise of a longitudinal force/unit length, related to the decay of load in the bar, and the force due to the radial stress $s(c)$. The use of the previously defined influence functions enables the longitudinal strain at $(0, 0, z)$ to be written as

$$\int_{L_1}^{L_2} \{q(c)K_{11}(0,0,z,R,c)+s(c)K_{12}(0,0,z,R,c)\}\,\mathrm{d}c-K_{11}(0,0,z,R,L_1)\,.\,p(L_1)$$
$$+K_{11}(0,0,z,R,L_2)\,.\,p(L_2) \tag{1}$$

where $p(L_1)$ and $p(L_2)$ are the end loads in the fictitious bar.

Consideration of the equilibrium of an elemental length of bar gives

$$q(c) = -\mathrm{d}p(c)/\mathrm{d}c \equiv -p'(c). \tag{2}$$

Thus the longitudinal strain in the medium at $(0, 0, z)$ is

$$-\int_{L_1}^{L_2} \{K_{11}(0,0,z,R,c)p'(c)-K_{12}(0,0,z,R,c)s(c)\}\,\mathrm{d}c-K_{11}(0,0,z,R,L_1)\,.\,p(L_1)$$
$$+K_{11}(0,0,z,R,L_2)\,.\,p(L_2)+K_{11}(0,0,z,R,L_3)\,.\,P^* \tag{3}$$

where P^* is a fraction of the externally applied force, P, and its magnitude is a function of the relative stiffnesses of bar and medium.

Similarly the radial strain at $(0, 0, z)$ is

$$-\int_{L_1}^{L_2} \{K_{21}(0,0,z,R,c)p'(c)-K_{22}(0,0,z,R,c)s(c)\}\,\mathrm{d}c-K_{21}(0,0,z,R,L_1)\,.\,p(L_1)$$
$$+K_{21}(0,0,z,R,L_2)\,.\,p(L_2)+K_{21}(0,0,z,R,L_3)\,.\,P^*. \tag{4}$$

With the assumption that the bar stresses in the fictitious bar are uniform across a bar diameter, and that for radial and longitudinal compatibility between the fictitious bar and the medium it suffices to equate strains along the line $x = y = 0$ in the medium and the bar centre line, then (3) and (4) give

$$p(z)/A_bE_f-2\nu_b s(z)/E_f = -\int_{L_1}^{L_2} \{K_{11}(0,0,z,R,c)p'(c)-K_{12}(0,0,z,R,c)s(c)\}\,\mathrm{d}c$$
$$-K_{11}(0,0,z,R,L_1)\,.\,p(L_1)+K_{11}(0,0,z,R,L_2)\,.\,p(L_2)$$
$$+K_{11}(0,0,z,R,L_3)\,.\,P^* \tag{5}$$

$$\nu_b p(z)/A_bE_f+(1-\nu_b)s(z)/E_f = \int_{L_1}^{L_2} \{K_{21}(0,0,z,R,c)p'(c)-K_{22}(0,0,z,R,c)s(c)\}\,\mathrm{d}c$$
$$-K_{21}(0,0,z,R,L_1)\,.\,p(L_1)+K_{21}(0,0,z,R,L_2)\,.\,p(L_2)$$
$$+K_{21}(0,0,z,R,L_3)\,.\,P^*. \tag{6}$$

As the load distribution in the fictitious bar may be discontinuous, the terms involving $p'(c)$ in equations (5) and (6) are inconvenient. Previous investigators of similar problems

[1, 3] have found that integration by parts eliminates this difficulty and equations (5) and (6) become

$$p(z)/A_bE_f - 2\nu_b s(z)/E_f = \int_{L_1}^{L_2} \{K'_{11}(0,0,z,R,c)p(c) + K_{12}(0,0,z,R,c)s(c)\}\,\mathrm{d}c$$
$$+ K_{11}(0,0,z,R,L_3)\,.\,P \tag{7}$$

$$-\nu_b p(z)/A_bE_f + (1-\nu_b)s(z)/E_f = \int_{L_1}^{L_2} \{K'_{21}(0,0,z,R,c)p(c) + K_{22}(0,0,z,R,c)s(c)\}\,\mathrm{d}c$$
$$+ K_{21}(0,0,z,R,L_3)\,.\,P. \tag{8}$$

It is noted that as a consequence of the integration by parts the term P^* in equations (3)–(6) becomes P, the externally applied force.

In equations (7) and (8)

$$K'_{11}(0,0,z,R,c) = \partial\{K_{11}(0,0,z,R,c)\}/\partial c$$
$$K'_{21}(0,0,z,R,c) = \partial\{K_{21}(0,0,z,R,c)\}/\partial c.$$

Equations (7) and (8) are integral equations for this problem. Henceforward, for brevity, functions of the form $f(0,0,z,R,c)$ will be written as $f(z,c)$, e.g.

$$K_{11}(z,c) \equiv K_{11}(0,0,z,R,c).$$

If the bar is treated one-dimensionally, neglecting transverse effects, then equation (7) becomes

$$p(z)/A_bE_f = \int_{L_1}^{L_2} \{K'_{11}(z,c)p(c)\,\mathrm{d}c\} + K_{11}(z,L_3)\,.\,P. \tag{9}$$

For the problem defined schematically in Fig. 1, with $P = 1$, equations (7) and (8) become

$$p(z)/A_bE_f - 2\nu_b s(z)/E_f = \int_{L_1}^{L_2} \{K'_{11}(z,c)p(c) + K_{12}(z,c)s(c)\}\,\mathrm{d}c + K_{11}(z,L_3) \tag{10}$$

$$-\nu_b p(z)/A_bE_f + (1-\nu_b)s(z)/E_f = \int_{L_1}^{L_2} K'_{21}(z,c)p(c) + K_{22}(z,c)s(c)\}\,\mathrm{d}c + K_{21}(z,L_3). \tag{11}$$

The required influence functions may be derived from a paper by Mindlin [5] and are given in Appendix A. The function $K'_{11}(z,c)$ has a singularity at $z = c$ arising from a finite discontinuity in the function $K_{11}(z,c)$.

If it is assumed that this discontinuity occurs in the range $(z-\mathrm{d}c) \le c \le (z+\mathrm{d}c)$ then the integral term involving $K'_{11}(z,c)$ in equation (10) becomes

$$\int_{L_1}^{L_2} K'_{11}(z,c)p(c)\,\mathrm{d}c = \underset{\mathrm{d}c\to 0}{Lt}\left\{\int_{L_1}^{z-\mathrm{d}c} K'_{11}(z,c)p(c)\,\mathrm{d}c + \int_{z-\mathrm{d}c}^{z+\mathrm{d}c} K'_{11}(z,c)p(c)\,\mathrm{d}c\right.$$
$$\left. + \int_{z+\mathrm{d}c}^{L_2} K'_{11}(z,c)p(c)\,\mathrm{d}c\right\}$$
$$= \int_{L_1}^{L_2} k'_{11}(z,c)p(c)\,\mathrm{d}c + \underset{\mathrm{d}c\to 0}{Lt}\int_{z-\mathrm{d}c}^{z+\mathrm{d}c} K'_{11}(z,c)p(c)\,\mathrm{d}c \tag{12}$$

where $k'_{11}(z, c)$ in the first integral term is continuous. Referring to Fig. 2 it may be seen that

$$\underset{dc \to 0}{Lt} \int_{z-dc}^{z+dc} K'_{11}(z, c)p(c)\,dc = -(1+\nu_m)(1-2\nu_m)p(z)/(1-\nu_m)\pi R^2 E_m.$$

Thus from equation (12)

$$\int_{L_1}^{L_2} K'_{11}(z, c)p(c)\,dc = \int_{L_1}^{L_2} k'_{11}(z, c)p(c)\,dc - (1+\nu_m)(1-2\nu_m)p(z)/(1-\nu_m)\pi R^2 E_m \quad (13)$$

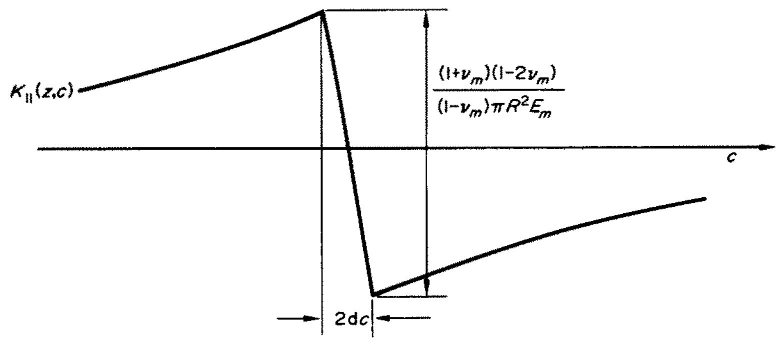

FIG. 2.

and equation (10) becomes,

$$p(z)/A_b E_f - 2\nu_b s(z)/E_f = \int_{L_1}^{L_2} \{k'_{11}(z, c)p(c) + K_{12}(z, c)s(c)\}\,dc$$
$$-(1+\nu_m)(1-2\nu_m)p(z)/(1-\nu_m)\pi R^2 E_m + K_{11}(z, L_3). \quad (14)$$

Equations (11) and (14) are the integral equations which apply to this problem.

If the bar is treated one-dimensionally a similar procedure to the above yields

$$p(z)/A_b E_f = \int_{L_1}^{L_2} k'_{11}(z, c)p(c)\,dc - (1+\nu_m)(1-2\nu_m)p(z)/(1-\nu_m)\pi R^2 E_m + K_{11}(z, L_3). \quad (15)$$

Equations (11) and (14) may be solved numerically for the force distribution, $p(z)$, in the fictitious bar. Hence the fraction of the externally applied load, P^*, and the decay of load in the fictitious bar are obtainable directly. The force distribution, $p_m(z)$, induced in the medium by these forces, P^*, $dp(z)/dc$, $p(L_1)$ and $p(L_2)$ may be derived. The actual force distribution in the bar, $p_b(z)$ is obtained by summation of the constituent contributions, $p(z)$ and $p_m(z)$.

3. RESULTS

The integral equation approach described previously has been adopted for the solution of a number of cases. Results for the particular cases of (a) a bar deeply embedded in the medium and axially loaded at its centre and (b) an end loaded bar projecting from the medium are presented in Figs. 3 and 4.

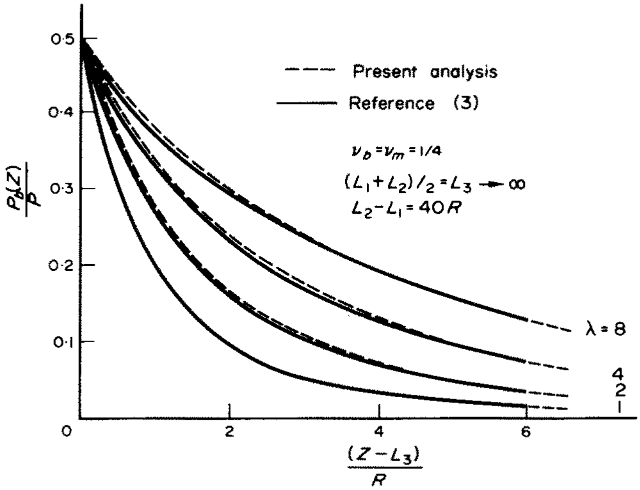

FIG. 3. Decay of load from a deeply embedded bar.

It was further felt desirable to obtain a solution using finite element techniques, for the projecting bar case, to permit consideration of the more realistic aspect of axial stress variation across the bar. Initial formulations incorporated elements of triangular cross section. These proved inadequate where high strain gradients prevailed and the results presented in Fig. 6 have been obtained from elements of quadrilateral cross section [6].

Figure 3 shows the results of the integral equation analysis as applied to a centrally loaded deeply embedded bar of length 40 bar radii; curves shown are based on an analysis including transverse effects. Results based on an analysis ignoring transverse effects differed by less than 1 per cent. When the bar is deeply embedded it may effectively be considered to be embedded within an infinite medium, and the results are directly comparable

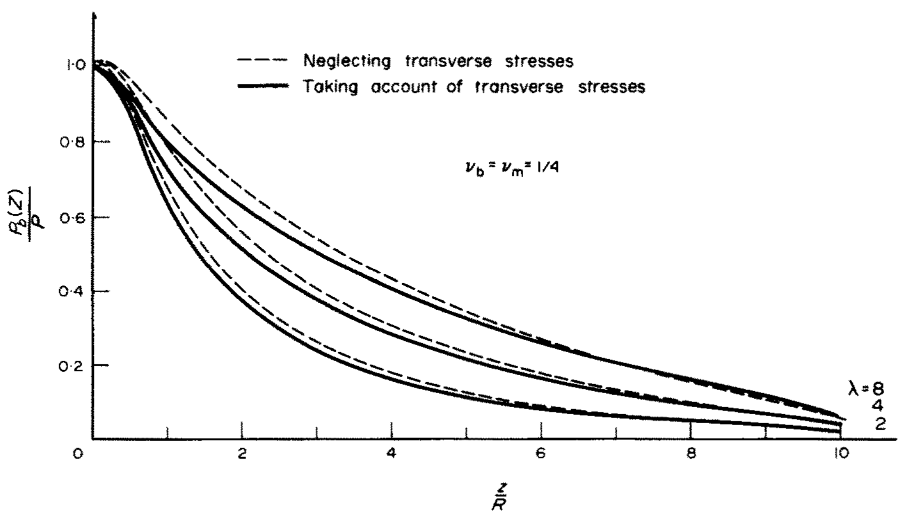

FIG. 4. Decay of load from a projecting bar, $L/R = 10$.

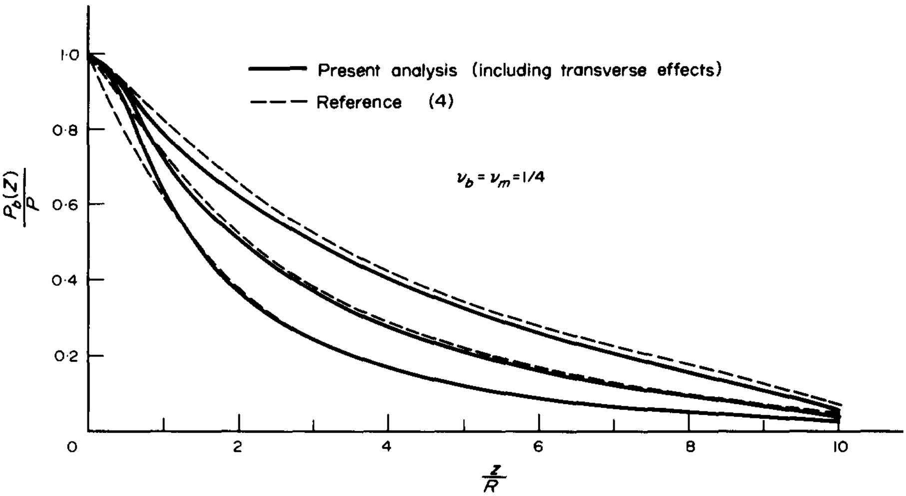

FIG. 5. Decay of load from a projecting bar, $L/R = 10$.

with those of Muki and Sternberg [3] who verified their numerical solution with an exact solution. Reference to Fig. 3 shows that the two sets of results are almost identical as one would expect due to the similarity of the analyses, although the line compatibility condition used by the present authors is rather simpler than the area compatibility condition.

Figure 4 compares the diffusion of load from an end loaded projecting bar, of length 10 bar radii, when transverse effects are considered with the solution obtained when transverse effects are not considered. An inadequacy of the line compatibility criterion is evident, near the origin, from the curves obtained neglecting transverse effects. The solution incorporating transverse effects is compared in Fig. 5 with the solution given in Ref. [4].

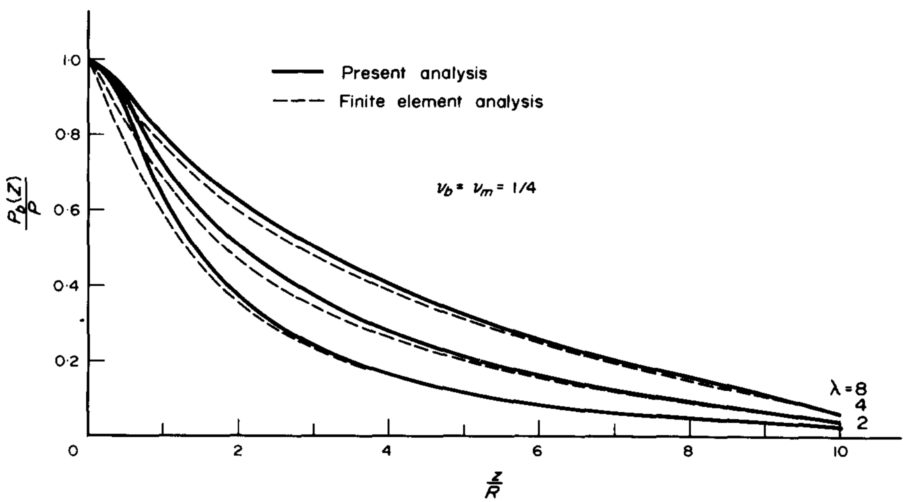

FIG. 6. Decay of load from a projecting bar, $L/R = 10$.

Figure 6 compares the integral equation solution with the finite element solution and is of interest in that the finite element solution permits a variation of axial stress across the bar diameter to be considered and this is of significance in the vicinity of the origin. However overall comparison is quite favourable. From the curves presented, for the lower stiffness ratios the area compatibility approach given in Ref. [4] compares more favourably with the finite element analyses. For the higher stiffness ratios the necessity of including transverse effects is evident.

4. CONCLUSIONS

An integral equation approach has been used to examine the load diffusion from an axially loaded bar embedded in a semi-infinite medium. Numerical results have been obtained for both projecting bars and deeply embedded bars. For embedded bars a one dimensional treatment of the bar is sufficient but for the projecting bar it is found that transverse stresses need be considered.

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APPENDIX

$$\begin{aligned}K_{11}(0,0,z,R,c) = \{&-(1-4\nu^2)(z-c)/R_1+(z-c)(-(z-c)^2+\nu(3R^2+2(z-c)^2))/R_1^3\\&+(1-2\nu)((z-c)+\nu(z+13c)-(3-4\nu)\nu(z+c))/R_2-((3-4\nu)z(z+c)^2\\&-c(z+c)(5z-c))/R_2^3-(4\nu c(z+c)((1-2\nu)(z+c)-c)\\&-\nu(3-4\nu)(z-c)(3R^2+2(z+c)^2))/R_2^3+cz(z+c)(-6(z+c)^2\\&+2\nu(5R^2+2(z+c)^2))/R_2^5+2(1-\nu)(1+n)\\&-4\nu(\nu n+1-2\nu^2)\}/4\pi R^2(1-\nu)E \qquad (A.1)\end{aligned}$$

$$\begin{aligned} K_{21}(0,0,z,R,c) = {} & \{2(1-2\nu)(z-c)/R_1 - 2(1-2\nu)(6c(1-\nu)+\nu(z-c) \\ & -(1-\nu)(1-2\nu)(z+c))/R_2 - (z-c)(3R^2(1-\nu)+2(z-c)^2(1-2\nu))/R_1^3 \\ & +2\nu(z+c)((3-4\nu)(z+c)z - c(5z-c))/R_2^3 + (1-\nu)(4c(z+c)((z+c) \\ & \times(1-2\nu)-c)-(3-4\nu)(z-c)(3R^2+2(z+c)^2))/R_2^3 \\ & -2cz(z+c)((1-\nu)(5R^2+2(z+c)^2)-6\nu(z+c)^2)/R_2^5 \\ & +4(1-2\nu)(1-\nu^2)\}/8\pi R^2(1-\nu)E \end{aligned} \tag{A.2}$$

$$\begin{aligned} K_{12}(0,0,z,R,c) = {} & R^2\,\mathrm{d}c\{(1-2\nu)(1+8\nu(1-\nu))/R_2^3 - (1-2\nu)/R_1^3 + 3((z-c)^2-\nu R^2)/R_1^5 \\ & +3(3-4\nu)((z+c)^2-\nu R^2)/R_2^5 - 6c(c+(1-2\nu)(z+c) \\ & +2\nu(z-c+(1-2\nu)(z+c)))/R_2^5 - 30cz((z+c)-\nu R^2)/R_2^7 \\ & +16(1-\nu)(1-2\nu)/R_2(R_2+z+c)^2 - 4\nu(1-\nu)(1-2\nu) \\ & \times(3R_2+z+c)R^2/R_2^3(R_2+z+c)^3\}/4(1-\nu)E \end{aligned} \tag{A.3}$$

$$\begin{aligned} K_{22}(0,0,z,R,c) = {} & R^2\,\mathrm{d}c\{2\nu(1-2\nu)/R_1^3 - 2(1-2\nu(\nu+4(1-\nu)^2)/R_2^3 + 3(R^2(1-\nu) \\ & -2\nu(z-c)^2)/R_1^5 + 3(3-4\nu)(R^2(1-\nu)-2\nu(z+c)^2)/R_2^5 \\ & +24cz(2-3\nu)/R_2^5 + 30cz(2\nu(z+c)^2-(1-\nu)R^2)/R_2^7 \\ & +16(1-\nu)^2(1-2\nu)/R_2(R_2+z+c)^2 - 4(1-\nu)^2(1-2\nu) \\ & \times(3R_2+z+c)R^2/R_2^3(R_2+z+c)^3\}/8(1-\nu)E \end{aligned} \tag{A.4}$$

with

$$\begin{aligned} R_1 &= +\sqrt{(R^2+(z-c)^2)} \\ R_2 &= +\sqrt{(R^2+(z+c)^2)} \\ n &= (z-c)/|z-c|. \end{aligned}$$

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Абстракт—Исследуется распределение течения нагрузки из конечного стержня, заделанного в полубесконечкой, упругой среде. Способ расчета при помощи интегрального уравнения дает возможность получить решение для сторжней полно цогруженных и с некоторой торчащей частью. Дается анализ для ряда случаев и получается сравнительные решения, пользуясь методом конечного злемента.